SUBGROUPS AND AUTOMORPHISMS OF EXTENDED SCHOTTKY TYPE GROUPS([1])

BY

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The purpose of this paper is to generalize some results on Schottky groups to extended Schottky type groups. Such groups represent compact Riemann surfaces with an even number of punctures.

The paper contains two results. In §3 we prove that all finitely generated subgroups of extended Schottky type groups are also extended Schottky type groups. This is a direct consequence of a theorem of Maskit's.

Since an extended Schottky type G is marked with a set of standard generators, the question arises which generators are standard for G. In §4 we prove that under an automorphism of G which preserves parabolic elements, standard generators map into standard generators. The proof relies on a theorem of automorphisms of free product groups of R. Prisco.

1. **Definitions and notation.** A Riemann surface S is said to be a *finite surface* with t *punctures* if it can be obtained from a compact Riemann surface by the removal of t points. A *retrosection* on a Riemann surface is an oriented simple closed curve. If α is a retrosection on the Riemann surface then a retrosection β is said to be *conjugate* to α if β intersects α in only one point and $\alpha \cdot \beta = 1$, i.e., the intersection number of α and β is one. We then say that (α, β) is a *retrosection pair*. A system of retrosections of type (r, s) on a finite Riemann surface S of genus $r+s$ with t punctures is a set of r retrosections $\gamma_1, \ldots, \gamma_r$ and s retrosection pairs $\lambda_1, \ldots, \lambda_s$ such that $S - \{\gamma_1, \ldots, \gamma_r, \lambda_1, \ldots, \lambda_s\}$ is homeomorphic to the $2r+s+t$ times punctured sphere.

Let G be a group of Möbius transformations viewed as automorphisms of the Riemann sphere P and $\Lambda(G)$ the set of limit points of G, i.e., $\Lambda(G)$ is the union of the derived sets of $\{Gz_0\}$ as z_0 ranges over P. We assume that G is *discontinuous*, i.e., $\Lambda(G) \neq P$. Then the factor space $S_G = (P - \Lambda(G))/G$ is a union of Riemann surfaces; the conformal structure of S_G is defined by requring that the natural map π of $P - \Lambda(G)$ onto S_G be holomorphic. A connected region (an open set together with part of its boundary) $D \subset P - \Lambda(G)$ is said to be a *fundamental region* for the discontinuous group G if $\bigcup_{g \in G} g(D) = P - \Lambda(G)$ and no two points of D are equivalent under G.

A group G of Möbius transformations is said to be a group of *extended Schottky type* (r, s, p) (or an E.S.T. group of type (r, s, p)) with *standard generators* $T_1, \ldots,$ $T_r, W_1, \ldots, W_p, U_1, V_1, \ldots, U_s, V_s$, and *defining curves* $\Gamma_1, \Gamma'_1, \ldots, \Gamma_r, \Gamma'_r, \Delta_1, \Delta'_1,$ $\ldots, \Delta_p, \Delta'_p, \Lambda_1, \ldots, \Lambda_s$ where Λ_i is a topological rectangle with sides $\Omega_i, \Sigma_i, \Omega'_i,$ $\Sigma'_i, i = 1, \ldots, s,$ if

(i) The curves are disjoint Jordan curves, except Δ_k, Δ'_k have a common point p_k, which bound a $2r + p + s$ times connected domain D such that $T_j(D) \cap D = W_k(D) \cap D = U_i(D) \cap D = V_i(D) \cap D = \varnothing$.

(ii) $T_j(\Gamma_j) = \Gamma'_j$, $W_k(\Delta_k) = \Delta'_k$, $U_i(\Omega_i) = \Omega'_i$, $V_i(\Sigma_i) = \Sigma'_i$, $j = 1, \ldots, r$; $k = 1, \ldots, p$; $i = 1, \ldots, s$. (See Figure 1.)

(iii) U_i and V_i are commuting parabolic transformations.

(iv) Group $\{U_i, V_i\}$ is discontinuous.

(v) W_k is parabolic with fixed point p_k.

We call $D \cup (\bigcup_{i=1}^r \Gamma_i) \cup (\bigcup_{k=1}^p \Delta_k) \cup (\bigcup_{j=1}^s \Omega_j \cup \Sigma_j)$ a *standard fundamental region* for G. Then $P - \Lambda(G)$ is connected and $(P - \Lambda(G))/G$ is a finite Riemann surface S_G of genus $r + s$ with $2p$ punctures, with a set of $r + s$ disjoint curves $\gamma_1, \ldots, \gamma_r,$ $\lambda_1, \ldots, \lambda_s$ distinguished on S_G such that $\gamma_1, \ldots, \gamma_r, \lambda_1, \ldots, \lambda_s$ is a complete system of retrosections of type (r, s). If $p = 0$ then G is said to be of Schottky type (r, s) and S_G is a compact surface.

Conversely, the fact that any finite Riemann surface of genus $r + s$ with $2p$ punctures and a complete system of retrosections of type (r, s) can be represented
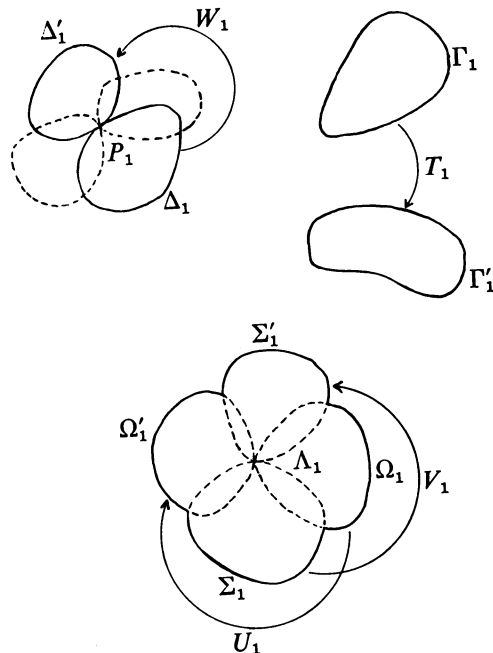

FIGURE 1. E.S.T. group of type $(1, 1, 1)$

by an E.S.T. group of type (r, s, p) is a classical theorem and can be proved by Bers' continuity method [3].

2. Properties of E.S.T. groups.

PROPERTY 1. Every E.S.T. group G of type (r, s, p) can be extended to a Schottky type group of type $(r, s+p)$.

Proof. Draw curves χ_k, χ'_k through p_k such that Δ_k, χ_k, Δ'_k, χ'_k "form" a topological rectangle. Let X_k be a parabolic transformation with fixed point p_k identifying χ_k and χ'_k such that the group $\{X_k, W_k\}$ is discontinuous. Then G is a subgroup of the group H generated by G and the X_k, $k=1, \ldots, p$, where H is of Schottky type $(r, s+p)$ (see Figure 1).

PROPERTY 2. If G is an E.S.T. group with standard generators $T_1, \ldots, T_r$, $W_1, \ldots, W_p, U_1, V_1, \ldots, U_s, V_s$, then G has the presentation $\langle T_1, \ldots, T_r, W_1, \ldots, W_p, U_1, V_1, \ldots, U_s, V_s; [U_j, V_j]=1, j=1, \ldots, s \rangle$, i.e., G is the free product of $r+p$ infinite cyclic groups and s free abelian groups on two generators.

Proof. The proof follows directly from Klein's Combination Theorem which states that in the formation of groups by combination no new relations are introduced [2, p. 209].

PROPERTY 3. If G is an E.S.T. group then G has no elliptic elements.

Proof. Any elliptic element belonging to G would have to be of finite order as G is discontinuous. But this is impossible as this would introduce a new relation in the combination group.

PROPERTY 4. An element belonging to an E.S.T. group G is parabolic iff it is conjugate in G to an element of the form W_k^n or $U_j^n V_j^m$ for some integers n and m.

We first prove a lemma. If Γ is a closed curve then Int Γ denotes the interior of Γ and Cl (Int Γ) the closure of Int Γ.

LEMMA 1. *Let G be an E.S.T. group with standard generators $T_1, \ldots, T_r, W_1, \ldots, W_p, U_1, V_1, \ldots, U_s, V_s$, and defining curves $\Gamma_1, \Gamma'_1, \ldots, \Gamma_s, \Gamma'_s, \Delta_1, \Delta'_1, \ldots, \Delta_p, \Delta'_p, \Lambda_1, \ldots, \Lambda_s$ bounding a domain D. Then if $W \in G$ and*

   (i) *$W=T_i^\alpha(\cdots)$, then*

      (a) *$W(D) \subset$ Int Γ'_i and W has a fixed point in Cl (Int Γ'_i) if $\alpha > 0$.*

      (b) *$W(D) \subseteq$ Int Γ_i and W has a fixed point in Cl (Int Γ_i) if $\alpha < 0$.*

   (ii) *$W=W_k^\alpha(\cdots)$, then*

      (a) *$W(D) \subset$ Int Δ'_k and W has a fixed point in Cl (Int Δ'_k) if $\alpha > 0$.*

      (b) *$W(D) \subset$ Int Δ_k and W has a fixed point in Cl (Int Δ_k) if $\alpha < 0$.*

   (iii) *$W=U_j^\alpha V_j^\beta(\cdots)$, $|\alpha|+|\beta| > 0$, then $W(D) \subset$ Int Λ_j and W has a fixed point in Cl (Int Λ_j), where we assume W is a freely reduced word in the generators of G.*

Proof. We prove only case (iii). The first half of the conclusion is just a special case of Klein's Combination Theorem. Now, using the fact that $W(D) \subset$ Int Λ_j, it is not difficult to check that $W(\text{Int } \Lambda_j) \subset$ Int Λ_j and thus conclude that W has a fixed point in Cl (Int Λ_j).

Proof of Property 4. Clearly all conjugates of $U_j^n V_j^m$ and W_k^n are parabolic. On the other hand if $A \in G$ is not conjugate to an element of this form, then without loss of generality we may assume A is not of this form. For example as a reduced word in the generators of G, A might be of the form $Z_1 T_i^{-1} U_j^n V_j^m Z_2$. Then a conjugate of A, $A' = U_j^n V_j^m Z_2 Z_1 T_i^{-1}$. By Lemma 1, A' has a fixed point in $Cl(Int \Lambda_j)$ and A'^{-1} and hence A' has a fixed point in $Cl(Int \Gamma_i')$. Thus A is loxodromic. The proof of all other cases is similar.

Property 4 is a special case of a more general theorem which we state as follows:

THEOREM 1. *In a combination group G satisfying the hypothesis of Klein's Combination Theorem an element is parabolic if and only if it is conjugate in G to a parabolic element belonging to one of the original groups.*

The proof of this theorem is essentially the same as that of Property 4.

3. **Subgroups of E.S.T. groups.** We say a domain $D \subset P$ is of *span zero* if the only univalent analytic functions defined on D are Möbius transformations. It is classical that if G is an E.S.T. group then $P - \Lambda(G)$ is of span zero (see Property 1 and [8]). On the other hand Maskit has proven that the property of $P - \Lambda(G)$ being of span zero essentially characterizes E.S.T. groups. Namely we have the following proposition:

PROPOSITION 2. *A discontinuous group is an E.S.T. group if and only if G is finitely generated, G has no elliptic elements and $P - \Lambda(G)$ is of span zero.*
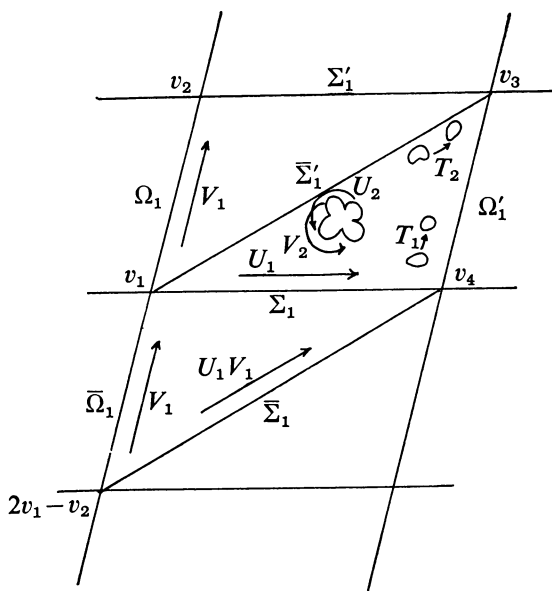

FIGURE 2. Automorphism of type $E_3(c)$

Outline of proof(^{2}). The only if part is clear. Conversely since $\Lambda(G)$ has span zero, $\Lambda(G)$ is connected, i.e., G has exactly one component $\Delta = \Lambda(G)$, and Δ is invariant under G. Let $S = \Delta/G$. Since G is finitely generated Ahlfors' Theorem [1] implies S is a finite surface. By the Planarity Theorem [6] the covering of S by Δ is given by a set of simple disjoint loops on S (G has no elements of finite order).

Starting with S and the set of loops construct (see [5]) a new group G^* with invariant component Δ^* which gives the same uniformization of S. By conformal invariance Δ^* also has span zero.

G^* is constructed as follows: Cut S along each of the loops and fill in the holes with discs. This gives a collection of finite surfaces $S_1, \ldots, S_k$. For each $i = 1, \ldots, k$ let G_i be the Fuchsian or elementary group representing S_i. Form the free product (Klein's Combination Theorem) of the G_i and the resulting group is a quasi-conformal deformation of G [5]. Since Δ^* has span zero, each G_i must be an elementary group. Thus G^* is formed via Klein's Combination Theorem as the free product of elementary groups without elliptic elements and we conclude that G is an E.S.T. group.

THEOREM 2. *Every finitely generated subgroup H of an E.S.T. group G is an E.S.T. group.*

Proof. Since H is a subgroup of G, $\Lambda(H) \subset \Lambda(G)$ and thus $P - \Lambda(H) \supset P - \Lambda(G)$ from which it follows that $P - \Lambda(H)$ is of span zero. The theorem now follows from Proposition 2.

4. **Automorphisms of E.S.T. groups.** We know that if G is an E.S.T. group of type (r, s, p) then there exists a standard set of generators for G such that

$$G = \langle T_1, \ldots, T_r, W_1, \ldots, W_p, U_1, V_1, \ldots, U_s, V_s; [U_j, V_j] = 1, j = 1, \ldots, s \rangle.$$

Under an automorphism of G which preserves the type of parabolic elements the standard generators go into other generators, $T_i \to T_i'$, $W_k \to W_k'$, $U_j \to U_j'$, $V_j \to V_j'$ such that in terms of these new generators G has the presentation

$$\langle T_1', \ldots, T_r', W_1', \ldots, W_k', U_1', V_1', \ldots, U_s', V_s'; [U_j', V_j'] = 1, j = 1, \ldots, s \rangle.$$

We prove in this section that the generators T_i', W_k', U_j', V_j' are also standard for G. To this end we examine the automorphisms of G.

AUTOMORPHISM THEOREM. *If an E.S.T. group*

$$G = \langle T_1, \ldots, T_r, W_1, \ldots, W_k, U_1, V_1, \ldots, U_s, V_s; [U_j, V_j] = 1, j = 1, \ldots, s \rangle$$

then every automorphism of G which preserves the type of parabolic elements can be obtained by finite repeated application of the following elementary automorphism:

(^{2}) The proof was communicated to me by B. Maskit.
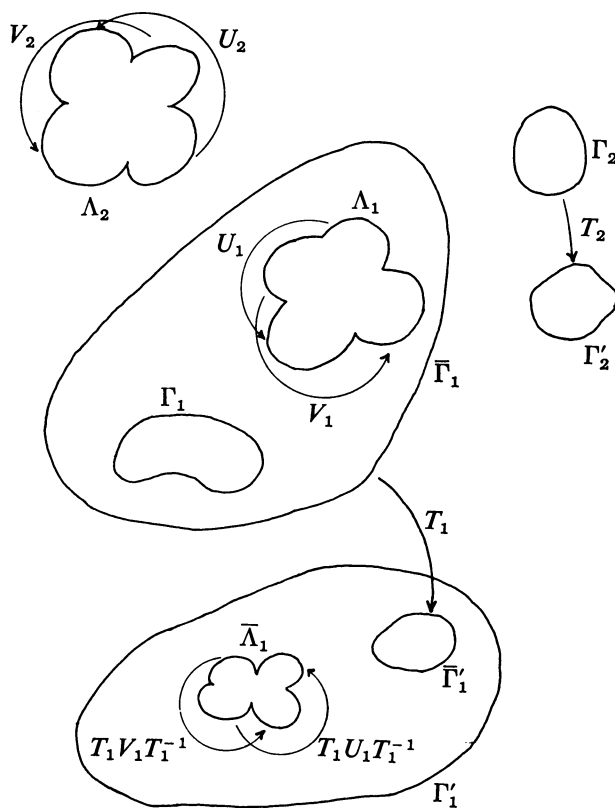
FIGURE 3. Automorphism of type $E_4(a)$

E_1(a) $T_j \rightarrow T_i$, $T_i \rightarrow T_j$, $i \neq j$ (*by this we mean map T_i into T_j, T_j into T_i and leave all other generators fixed*); (b) $T_i \rightarrow T_i^{-1}$; (c) $T_i \rightarrow T_k T_i$, $i \neq k$.

E_2(a) $W_j \rightarrow W_i$, $W_i \rightarrow W_j$, $i \neq j$; (b) $W_i \rightarrow W_i^{-1}$; (c) $W_i \rightarrow W_k W_i$, $i \neq k$.

E_3(a) $U_i \rightarrow V_j$, $V_j \rightarrow U_j$; (b) $U_j \rightarrow U_j^{-1}$; (c) $U_i \rightarrow U_i V_i$.

E_4(a) $U_j \rightarrow T_i U_j T_i^{-1}$, $V_j \rightarrow T_i V_j T_i^{-1}$; (b) $U_j \rightarrow W_i U_j W_i^{-1}$, $V_j \rightarrow W_i V_j W_i^{-1}$; (c) $U_j \rightarrow V_i U_j V_i^{-1}$, $V_j \rightarrow V_i V_j V_i^{-1}$, $i \neq j$.

E_5(a) $T_i \rightarrow V_j T_i$; (b) $T_i \rightarrow W_j T_i$.

E_6 $U_j \rightarrow U_k$, $U_k \rightarrow U_j$, $V_j \rightarrow V_k$, $V_k \rightarrow V_j$, $j \neq k$.

For a proof of this theorem see [7].

We show that under the elementary automorphisms E_1–E_6 standard generators for G go into standard generators. Certain cases, namely E_1(a), E_1(b), E_2(a), E_2(b), E_3(a), E_3(b), E_6 are trivial. For the following lemmas we assume that G is an E.S.T. group with presentation as in the preceding theorem, and that those generators are standard for G.

LEMMA 2. *Automorphisms of type* $E_1(c)$ *preserve the standard generators of G.*

Proof. The proof is an exact repetition of that of Lemma 1 of [4].

LEMMA 3. *Automorphisms of type* $E_3(c)$ *preserve the standard generators of G.*

Proof. We assume $i=1$ and that the common fixed point p of U_1, V_1 is ∞. Let the vertices of the topological rectangle $\Lambda_1 = \Omega_1 \Sigma_1 \Omega_1' \Sigma_1'$ be v_1, v_2, v_3, v_4 so that $U_1(v_1) = v_4$, $U_1(v_2) = v_3$, $V_1(v_1) = v_2$, and $V_1(v_4) = v_3$. We draw a "diagonal" $\overline{\Sigma}_1'$ connecting v_1 and v_3 in such a way that $\overline{\Sigma}_1' \subset \text{Int } \Lambda_1$ and the topological triangle $\Omega_1' \Sigma_1 \overline{\Sigma}_1'$ contains in its interior all the other defining curves of G (see Figure 2). Let $\overline{\Sigma}_1 = V_1^{-1}(\overline{\Sigma}_1')$, $\overline{\Omega}_1 = V_1^{-1}(\Omega_1)$ and $\overline{\Lambda}_1 = \overline{\Omega}_1 \overline{\Sigma}_1 \Omega_1' \overline{\Sigma}_1'$. It is then easily verified that the generators $T_1, \ldots, T_r, W_1, \ldots, W_k, U_1 V_1, V_1, U_2, V_2, \ldots, U_s, V_s$, are standard generators for G with defining curves $\Gamma_1, \Gamma_1', \ldots, \Gamma_r, \Gamma_r', \Delta_1, \Delta_1', \ldots, \Delta_k, \Delta_k', \overline{\Lambda}_1, \Lambda_2, \ldots, \Lambda_s$.

LEMMA 4. *Automorphisms of type* $E_4(a)$ *preserve the standard generators of G.*
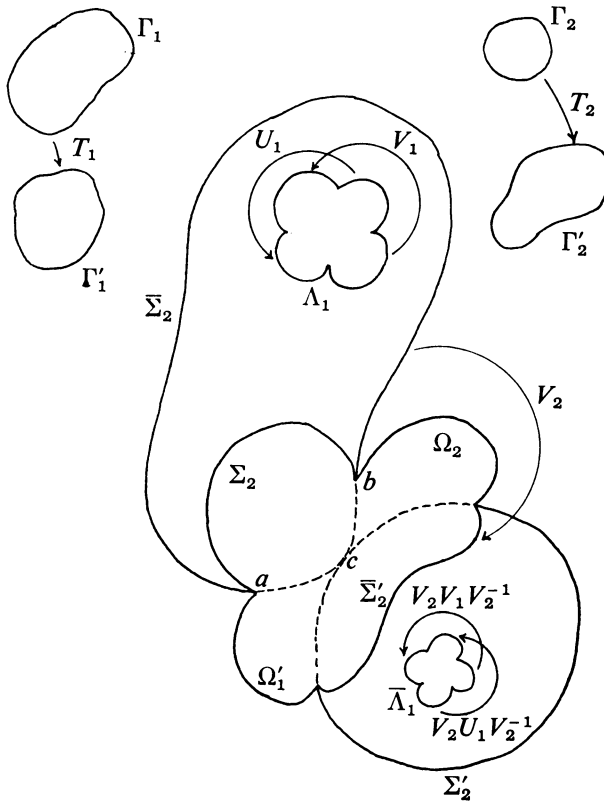

FIGURE 4. Automorphism of type $E_4(c)$

Proof. Figure 3 suffices to indicate the outline of the proof. Here $i=j=1$. The details of the proof are omitted as they are similar to the previous argument.

LEMMA 5. *Automorphisms of type* $E_4(c)$ *preserve the standard generators of G.*

Proof. See Figure 4. Here $i=2$, $j=1$.

LEMMA 6. *Automorphisms of type* $E_5(a)$ *preserve the standard generators of G.*

Proof. See Figure 5. Here $i=1$, $j=2$.
We omit the lemmas for automorphisms of type $E_2(c)$, $E_4(b)$, and $E_5(b)$ as they are similar to the preceding ones.

THEOREM 3. *If G is an E.S.T. group then under any automorphism of G preserving parabolic elements, standard generators are mapped onto standard generators.*

Proof. The proof follows directly from the preceding lemmas and the automorphism theorem.
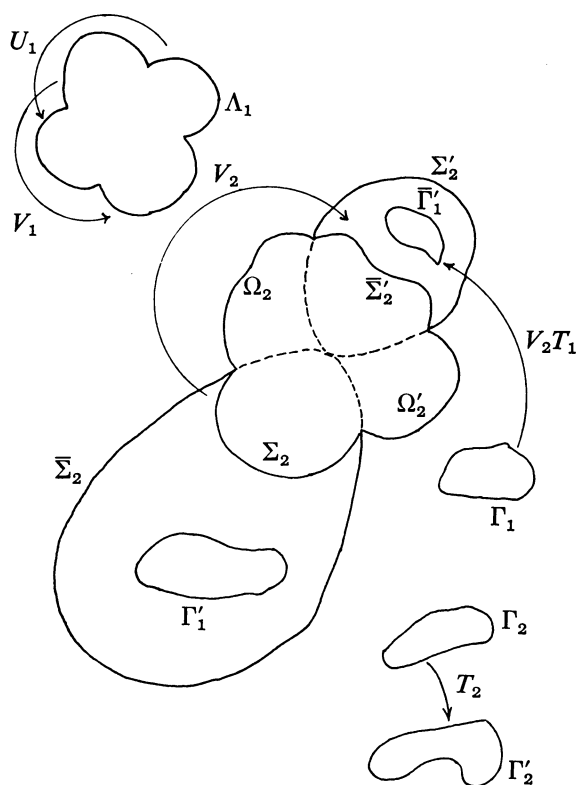

FIGURE 5. Automorphism of type $E_5(a)$

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